

IB GROUPS, RINGS AND MODULES

Lent Term 2023

Example Sheet 1 of 4

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- (1)
 - (a) What are the orders of elements of the group S_4 ? How many elements are there of each order?
 - (b) How many subgroups of order 2 are there in S_4 ? Of order 3? How many cyclic subgroups are there of order 4?
 - (c) Find a non-cyclic subgroup $V \leq S_4$ of order 4. How many such subgroups are there?
 - (d) Find a subgroup $D \leq S_4$ of order 8. How many such subgroups are there?
- (2)
 - (a) Show that A_4 has no subgroups of index 2. Exhibit a subgroup of index 3.
 - (b) Show that A_5 has no subgroups of index 2, 3, or 4. Exhibit a subgroup of index 5.
 - (c) Show that A_5 is generated by $(12)(34)$ and (135) .
- (3) Calculate the size of the conjugacy class of (123) as an element of S_4 , as an element of S_5 , and as an element of S_6 . Find in each case its centraliser. Hence calculate the size of the conjugacy class of (123) in A_4 , in A_5 , and in A_6 .
- (4) Suppose that $H, K \triangleleft G$ with $H \cap K = 1$. Show that any element of H commutes with any element of K . Hence show that $HK \cong H \times K$.
- (5) Let p be a prime number, and G be a non-abelian group of order p^3 .
 - (a) Show that the centre $Z(G)$ of G has order p .
 - (b) Show that if $g \notin Z(G)$ then its centraliser $C(g)$ has order p^2 .
 - (c) Hence determine the sizes and numbers of conjugacy classes in G .
- (6)
 - (a) For $p = 2, 3$ find a Sylow p -subgroup of S_4 , and find its normaliser.
 - (b) For $p = 2, 3, 5$ find a Sylow p -subgroup of A_5 , and find its normaliser.
- (7) Show that there are no simple groups of orders 441 or 351.
- (8) Let p, q , and r be prime numbers, not necessarily distinct. Show that no group of order pq is simple. Show that no group of order pq^2 is simple. Show that no group of order pqr is simple.
- (9)
 - (a) Show that any group of order 15 is cyclic.
 - (b) Show that any group of order 30 has a normal subgroup of order 15.
- (10) (Semi-direct product) Let N and H be groups, and $\phi : H \rightarrow \text{Aut}(N)$ a homomorphism. Show that we can define a group operation on the set $N \times H$ by

$$(n_1, h_1) \bullet (n_2, h_2) = (n_1 \cdot \phi(h_1)(n_2), h_1 \cdot h_2).$$

Show that the resulting group G contains copies of N and H as subgroups, that N is normal in G , that $NH = G$, and that $N \cap H = 1$.

By finding an element of order 3 in $\text{Aut}(C_7)$, construct a non-abelian group of order 21.

Further Questions

- (11) Let p be a prime number. How many elements of order p are there in S_p ? What are their centralisers? How many Sylow p -subgroups are there? What are the orders of their normalisers? If q is another prime number which divides $p - 1$, show that there exists a non-abelian group of order pq .
- (12) Show that there are no simple groups of order 300 or 112.
- (13) Show that a group G of order 1001 contains normal subgroups of order 7, 11, and 13. Hence show that G is cyclic. [Hint: You may want to use Question 4.]
- (14) Let G be a simple group of order 60. Deduce that $G \cong A_5$, as follows. Show that G has six Sylow 5-subgroups. By considering the conjugation action on the set of Sylow 5-subgroups, show that G is isomorphic to a subgroup $G \leq A_6$ of index 6. By considering the action of A_6 on A_6/G , show that there is an automorphism of A_6 taking G to A_5 .
- (15) Let G be a group of order 60 which has more than one Sylow 5-subgroup. Show that G is simple.
- (16) Let G be a finite group with cyclic and non-trivial Sylow 2-subgroup. By considering the permutation representation of G on itself, show that G has a normal subgroup of index 2. [Hint: Show that a generator for the Sylow subgroup induces an odd permutation of G .]
- (17) (Frattini argument) Let $K \triangleleft G$ and P be a Sylow p -subgroup of K . Show that any element $g \in G$ may be written as $g = nk$ with $n \in N_G(P)$ and $k \in K$, and hence that $G = N_G(P)K$. [Hint: Observe that P and $g^{-1}Pg$ are both Sylow p -subgroups of K .] Deduce that $G/K \cong N_G(P)/N_K(P)$.

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